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Momentum, Pressure, and Energy in the Trefftz-Plane

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The distributions of momentum and energy generated by a force working in a fluid together with the associated pressure distributions are discussed in detail. The momentum generated by an impulsive point force in a compressible atmosphere is identified for two-thirds in a moving virtual sphere of fluid and for one-third in a front expanding with the velocity of sound. In a transverse section of an incompressible atmosphere the net momentum is identically zero when boundary conditions are left out of the considerations. It is discussed where and how net momentum is imparted to the atmosphere by impulsive and steady, moving force distributions. The analogy between an actuator surface and a viscous boundary layer on a flat plate, both involving the generation of momentum in a definite jet or wake region, is discussed in the same frame of reference as the induced drag of a lifting wing. The boundary conditions on and the dimensions of the control box both play a role in the answer to the question if and where net momentum is to be found. The statement that the net momentum in a transverse section remains zero proves to be only approximately valid in a free atmosphere. An important conclusion is that at an overall scale the unsteady pressures in any model, viscous or nonviscous, dominate the momentum balance.

Nomenclature

A	=	cross-sectional area, m^2
a	=	radius of cylinder or sphere, m
b	=	span of wing or distance between singular vortices, m
C	=	control volume
c	=	speed of sound
D	=	drag force, kg m/s^2
$\mathbf{F}$, F	=	force, kg m/s^2
f	=	factor, $0 < f < 1$
G	=	integral of g
g	=	dummy variable
H	=	Heaviside unit step function
k	=	factor, $0 < k < 2$
L	=	lift force, N
l_1	=	x dimension of control volume, m
l_2	=	x dimension of rear control volume, m
$\mathbf{M}$, M	=	momentum, kg m/s
m	=	mass, kg or m
N	=	integer number or Newton, kg m/s^2
$\mathbf{n}$	=	unit-normal vector
$\mathbf{P}$, P	=	impulse, N s
p	=	pressure, N/m^2
q	=	source strength, m^2/s or m^3/s
R, ϕ, θ	=	spherical coordinates
R_0	=	radial distance between observer and singularity, m
r, ϕ, z	=	cylinder coordinates
S	=	bounding surface of a volume, m^2
s	=	second, s
T	=	force, N
t	=	time, s
U	=	velocity of moving force
u, v, w	=	x-, y-, z-velocity components in the field
V	=	volume
w''	=	velocity component in system inclined at α'', m/s
X, Y	=	dimensions of control box, m
x, y, z	=	Cartesian coordinates, m
α''	=	downward angle of rolled-up trailing vortex system
α_i	=	downward angle of flat trailing vortex system near wing
Γ	=	strength of singular vortex, m^2/s
γ	=	strength of vortex distribution per unit of length, m/s
δ	=	boundary-layer thickness, m
δ^*	=	displacement thickness of boundary layer, m
θ	=	momentum thickness of boundary layer, m
ρ	=	mass density, kg/m^3
Φ	=	velocity potential function, m^2/s

Subscripts

aft	=	aft face of total control volume
air	=	applied to the air in contrast to applied to device
fore	=	foreface of total control volume
in	=	directed inward
L	=	left
out	=	directed outward
R	=	right

Introduction

THIS article contains a detailed discussion of the generally well-known effects of a force working in a fluid, i.e., distributions of momentum and energy. A compact force field, yielding a nonzero net force, necessarily is rotational, and therefore, it generates vorticity. According to Kelvin's theorem in a nonviscous fluid the generated vorticity will be convected with the fluid. In a viscous fluid the vorticity will be transported by convection and distributed by diffusion. As a consequence of Newton's law the force field imparts momentum and as the force performs work it delivers energy to the fluid. The impulse of a vortex system is defined, following Kelvin (see Ref. 1 or Ref. 8), as the externally applied impulsive force that would generate the system in infinite space; the resulting momentum content of a volume depends on the total force on that volume, including the pressure forces on the boundaries and other bodies inside the volume.

The aim of this article is to bring the different mechanisms by which forces can be exerted on the atmosphere under the same heading of unsteady pressure forces. This is done by comparing the role of the terms in the momentum balance

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describing various model situations. The anologies will be discussed in a reference frame fixed to an atmosphere at rest at infinity. This point of view requires an unsteady description, thereby avoiding the pitfall of considering things to be steady that are not. As an example may serve the vortex sheet at the edge of a jet. For the kinematic description it does not matter whether the vortices move or not, for the description of the pressure field the motion is of prime importance.

In the model situations to be treated a first group consists of elementary situations involving moving singularities generated by impulsive forces. They are as follows:

1) A three-dimensional impulsive point force in a nonviscous compressible fluid generating a ring vortex: in a description in terms of a potential it is a moving dipole. A spherical body of fluid moves with the dipole, the sphere contains two-thirds of the momentum. The wave front carries away one-third of the momentum.

2) A two-dimensional impulsive point force in a nonviscous fluid generating a vortex couple or a two-dimensional dipole: a circular cylinder moves with the dipole as a virtual body of fluid. It is shown[2] that the momentum content of a control volume depends on the shape of the volume as well as on the boundary conditions, which are eventually to be applied outside the control volume. The momentum distribution associated with a dipole singularity (equivalent to a vortex couple) shows singular behavior in the sense that the total impulse is present as momentum in the infinitely narrow "strip" containing the axis of the dipole.

3) A two-dimensional impulsive force distribution of constant magnitude on a straight line (impulsive actuator line) in a nonviscous fluid generating a pair of vortices: an oval body of fluid moves with the vortices. The total impulse is present as momentum in the infinite strip between the vortices.

4) A flat plate moving normal to its plane: it is not accompanied by a virtual body of fluid.

A second group treats steady forces generating a trailing vortex sytem. The forces move steadily in a fixed frame of reference. The models concern the following:

1) A two-dimensional actuator surface moving in a nonviscous fluid generating a slipstream containing the momentum increase: the boundary between the slipstream and the surrounding fluid consists of two moving vortex sheets.

2) A flat plate moving in a viscous fluid generating a vorticity distribution in viscous boundary layers: this model may be looked upon as a viscous equivalent of the actuator surface. The blockage effect is modeled by a moving source, analogous to the actuator surface model. In both flow configurations the force vectors are parallel to the motion vector.

3) A lifting wing moving in a nonviscous fluid generating a trailing vortex system: the lift force (perpendicular to the motion vector of the wing) is accompanied by a drag force along the motion vector. The trailing vortex system is slanted downward behind the wing. The model considers a trailing vortex sheet rolling up to form a pair of vortices, closed by a starting vortex. A blockage effect is identified, related to the rate of extension of the trailing vortex system. The vertical momentum in the fluid necessarily is accompanied by horizonal momentum. The horizontal and vertical boundary conditions on the atmosphere are independent. Considering a towed aircraft results in accelerating the atmosphere in the direction of the motion, whereas the net vertical momentum remains zero. It is possible though to identify parts of the atmosphere that contain the entire lift as vertical momentum. Analogously it is possible to identify volumes around the Trefftz-plane that contain part of the drag as momentum, associated with regions of higher pressure. The net dynamic pressure integral over the Trefftz-plane would overestimate the induced drag as there is the momentum transport term in the Trefftz-plane. It is only on a control volume C_{total} so large that quadratic velocity disturbances are negligibly small on the end faces that the unsteady pressure integrals balance the drag. The pressure on the end faces is governed by the boundary conditions to be applied there.

Reynolds' Transport Theorem

In order to avoid ambiguity, Reynolds' transport theorem is recapitulated. Control volumes are defined for reference in the following section.

Newton's law, $\boldsymbol{F} = m\, \mathrm{d}\boldsymbol{v}/\mathrm{d}t$, will be applied to a moving and deforming mass (of air) occupying a volume $V(t)$ at the moment of observation. Newton's law then reads

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t} \int\int_{V(t)}\int \rho \boldsymbol{v}\, \mathrm{d}V \tag{1}$$

The difficulty of working with a deforming volume integral is handled by the use of Reynolds' transport theorem.[3] The theorem relates the material time derivative of such a moving and deforming volume integral to the transport of the integrand through the boundaries S_0 of the volume V at the moment t and the partial time derivative of the integrand inside the volume V_0, at t. Reynolds' transport theorem applied to a volume integral $\boldsymbol{G}$ of a function $\boldsymbol{g}$ reads

$$\frac{\mathrm{d}\boldsymbol{G}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int\int_{V(t)}\int \boldsymbol{g}\, \mathrm{d}V = \int\int_{V_0}\int \frac{\partial \boldsymbol{g}}{\partial t}\, \mathrm{d}V_0 + \int_{S_0}\int \boldsymbol{g}(\boldsymbol{v}\cdot\boldsymbol{n})\, \mathrm{d}S_0 \tag{2}$$

Equation (2) is the integral form of the relation between the Lagrangian and the Eulerian description of fluid motion.

In the following considerations $\boldsymbol{G}$ will be either the scalar mass m or the momentum vector $\boldsymbol{M}$. Newton's law [Eq. (1)], applied to the mass inside a control volume at the time t thus becomes

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \int\int_{V_0}\int \frac{\partial(\rho\boldsymbol{v})}{\partial t}\, \mathrm{d}V_0 + \int_{S_0}\int (\boldsymbol{v}\cdot\boldsymbol{n})\rho\boldsymbol{v}\, \mathrm{d}S_0 \tag{3}$$

The form (3) will be referred to as the momentum balance of the control volume V_0.

In the following sections the net force $\boldsymbol{F}$ will be composed of the externally applied force and the pressure integrals on the faces of the volume V_0 (referred to as the control volume C). The right-hand side (RHS) of (3) contains a volume integral describing the rate of change of the momentum content of the volume V_0, and an area integral taking into account the transport of momentum through its bounding areas.

A peculiar property of the surface integral is worth mentioning. The momentum transport through an area normal to the velocity vector is identically positive (negative) when the normal is in a positive (negative) coordinate direction. The example situation shown in Fig. 1 may illustrate this remarkable result. Some mechanism inside the (at one side open) rigid container C generates the velocity distribution as shown. There is no net outflow through the area A, so that v_{in} =

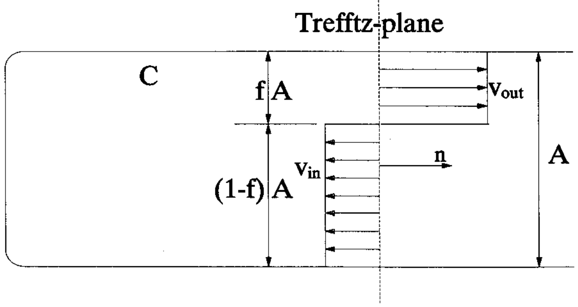

Fig. 1 Normal-velocity distribution in the Trefftz-plane yields an identically positive contribution to the momentum transport.

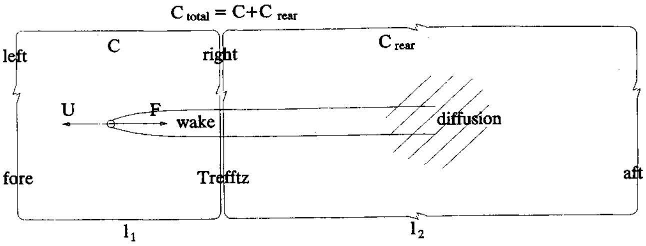

Fig. 2 Definition of the control volumes C, C_{rear}, and C_{total}.

$-v_{out}f/(1-f)$. The contribution of this velocity-distribution to the momentum transport integral then yields the identically positive amount

$$\begin{aligned} dM_{transport} &= \rho A[(1-f)(v_{in}\cdot n)v_{in} + f(v_{out}\cdot n)v_{out}] \\ &= \rho A[(1-f)v_{in}^2 + fv_{out}^2] = \rho Afv_{out}[(-v_{in}+v_{out})\cdot n] \\ &= \rho Afv_{out}(|v_{in}| + |v_{out}|) \end{aligned} \tag{4}$$

The momentum balance will be applied to a control volume C fixed to the atmosphere at rest. The vertical faces are identified by the subscripts left and right (see Fig. 2). In the models of the actuator surface, the viscous boundary layer, and the lifting wing there is a wake containing the generated vortices. The wake is supposed to intersect one of the vertical bounding areas of the control volume at a location where the horizontal gradients have a much smaller order of magnitude than the vertical gradients. The intersecting plane is also referred to as the Trefftz-plane. The vorticity in the wake finally diffuses in the volume C_{rear} next to the Trefftz-plane. A control volume C_{total} built up of C and C_{rear} (Fig. 2) is supposed to contain the whole atmosphere. Its end faces will be referred to as fore and aft faces. Boundary conditions are applied on the fore and aft area. The homogeneous pressure field that must be added to satisfy the boundary conditions is supposed to generate a pressure gradient of the order $\mathcal{O}[(F/\text{Area})/(l_1 + l_2)]$. The horizontal proportions of the control volumes C and C_{rear} are chosen in such a way that the associated pressure forces on the left and right areas of C are negligible in relation to those on the front and rear areas of C_{total}. That means that under the condition $l_1 << l_2$ the effect of the boundary conditions on A_{fore} and A_{aft} is negligible in the momentum balance of C.

Elementary Singularities Generated by Impulsive Force Distributions in a Nonviscous Fluid

Formation of an Elementary Three-Dimensional Ring Vortex in a Compressible Medium, Impulsive Motion of a Sphere

An impulsive point force $F\delta(t-t_0)\delta(x-x_0)$ per unit mass, (in x direction for simplicity) generates a singular moving ring vortex. The velocity field is described by the potential of a moving dipole, switched on by the unit step function $H(t-t_0)$, which is the time integral of $\delta(t-t_0)$. In a compressible fluid the field expands spherically with the velocity of sound c. The unsteady field is described by the potential

$$\begin{aligned} \Phi(x,y,z,t) &= -\frac{(x-Ut_0)\delta(t-t_0-R_0/c)P}{4\pi c\tilde{R}R_0} \\ &\quad -\frac{(x-Ut)H(t-t_0-R_0/c)P}{2\pi\tilde{R}^3} \\ \tilde{R} &= \{(x-Ut)^2 + \beta^2[(y-y_0)^2 + (z-z_0)^2]\}^{1/2} \\ \beta^2 &= (1-U^2/c^2) \\ R_0 &= [(x-Ut_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2} \end{aligned} \tag{5}$$

The dipole strength P is equal to F/ρ. Rejecting the idea that the ring vortex is moving force-free with its infinite self-induced velocity, there is freedom to choose the magnitude of the velocity U of the dipole. When considering the streamline pattern in an incompressible fluid in a coordinate system moving with the dipole, a spherical stream surface of radius a is to be observed, where a is determined from the relation $P = 2\pi Ua^3$. For the velocity field outside this sphere it does not matter whether this is the surface of a material sphere or just the stream surface in the field of a dipole. Considering the ratio U/c to be very small, the compressible expression (5) approximately represents the flow potential of a sphere of radius a set into motion at $t = t_0$ with velocity U. The kinetic energy of the dipole flow inside the spherical stream surface would be infinite. Therefore, the force is supposed to be applied to a solid sphere of density ρ, the kinetic energy of the sphere then is $(4/3)\pi a^3\rho U^2/2$ and its momentum is $(4/3)\pi a^3\rho U$. This momentum is different from the impulsive force $F(=2\pi\rho Ua^3)$. When no other forces have been working, the momentum balance requires all of the impulse to be converted to momentum in the fluid. In a compressible fluid the point force generates a spherical wave [Eq. (5)]. In the model with the solid sphere this wave is generated by the impulsive pressures on the surface of the sphere, setting up the velocity field of the moving dipole between the front and the sphere. During the time that this front has not yet encountered any boundaries, the total impulse is present as momentum in the fluid.

It so proves that two-thirds of the impulse result in momentum of the sphere, the remaining one-third is found as momentum in the expanding front of the unit step wave. Integrating $\rho\partial\Phi/\partial x$ over the spherical wave and the front yields as a result that the δ-function front in (5) does not contain momentum. The momentum in the front of the step function is

$$\begin{aligned} \int_{front}\int \rho\frac{\partial\Phi}{\partial x}\,dA &= \int_0^\infty dR_0 \\ &\times \int_0^\pi \frac{2\pi\rho Ua^3(x-Ut)(x-Ut_0)\delta(t-t_0-R_0/c)}{4\pi\tilde{R}^3R_0c} \\ &\times 2\pi R_0^2\sin\theta\,d\theta \approx \pi\rho Ua^3\int_0^\infty dR_0\delta[R_0 - c(t-t_0)] \\ &\times \int_0^\pi \cos^2\theta\sin\theta\,d\theta = \frac{2}{3}\pi\rho Ua^3 \end{aligned} \tag{6}$$

The approximate sign is used because the compressible potential, involving $\tilde{R}$, is approximated using $R_0 \approx \tilde{R}$. The sum of the momenta in the front and in the sphere equals the impulse P. The volume between the sphere and the front does not contain a net momentum.

As soon as the front hits a solid boundary, e.g., a plane wall perpendicular to the motion vector, the front will be reflected. After a long time the net momentum in the system of primary and reflected fronts will vanish. The half of the spherical front moving away from the wall from the outset contains positive momentum, the spherical cap forming the reflected front will finally contain an equal amount of negative momentum. The "reflected" momentum in the space bounded by the spherical cap and the wall amounts to $[(1-\cos^3\theta_0)/6 - (1-\cos\theta_0)/2]$, where θ_0 is the angle spanned by the cap. The net momentum normal to the wall of the half-space gradually vanishes while the dipole approaches the wall.

In an incompressible atmosphere the impulsive force is instantaneously accompanied by an impulsive pressure field. The acceleration is a delta-function in time and the resulting velocity U grows as a step function in time. The impulsive force $F\delta(t-t_0)$ would, if it were the only force working, impart a momentum $\int F\delta(t-t_0)\,dt$ to the system of solid sphere and fluid. The momentum of the moving solid sphere is only $2\pi\rho Ua^2/3$. The net force on the control volume has,

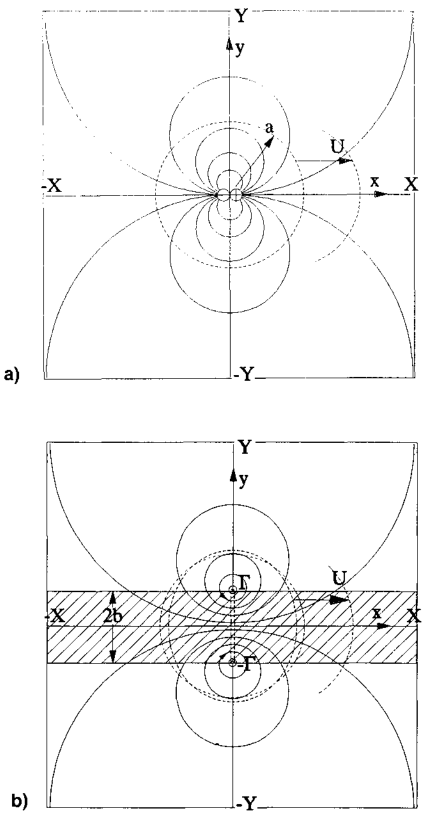

Fig. 3 a) Dipole with impulse $P = 2\pi Ua^2$ and b) a vortex couple with impulse $P = 2b\Gamma$.

next to the force $F\delta(t - t_0)$ on the sphere, contributions from pressure forces on the boundaries. The role of the associated pressure integrals is discussed in the next section on the (simpler) unsteady incompressible two-dimensional problem.

Momentum in the Two-Dimensional Field of a Moving Dipole

An impulsive point force $F\delta(t - t_0)$ (in x direction for simplicity) generates a singular moving vortex couple. The velocity field is described by the potential of a moving dipole of strength $P = F/\rho$, switched on by the unit step function $H(t - t_0)$

$$\Phi(x, y, t)H(t - t_0) = \frac{-Ua^2[x - U(t - t_0)]}{[x - U(t - t_0)]^2 + y^2} H(t - t_0)$$
$$P = F/\rho = 2\pi a^2 U \qquad (7)$$

The singularities are set into motion with velocity U in an atmosphere at rest (Fig. 3a).

The momentum balance is applied to a rectangular control box with sides $2X$, $2Y$. This control box is situated inside a much larger container (an example of such a container is the atmosphere) at the boundaries (not necessarily walls) of which boundary conditions are to be applied. The net force on the mass in the control box is built up of the sum of F and the unsteady pressure on the boundary of the control box. The variation of the momentum is composed of the variation inside the box and the transport through the boundaries. The momentum inside the box is computed from integration of $\partial\Phi/\partial x$ over the cross section in the x-y plane. The net mass transport through the boundaries must vanish. The pressure integrals over the boundaries normal to the motion formally depend on the boundary conditions on the container. When the container is free-suspended, like the atmosphere is horizontally as fore and aft "walls" are connected, the pressure integral on the boundary of the container must be zero. When the container is fixed to the coordinate system, like the atmosphere is vertically, the net force on the boundary of the container must balance the applied force F, yielding a net zero momentum inside the container. When the control box is much smaller than the container the pressure gradient created by the boundary conditions on the container induces negligible effects on the boundaries of the control box. Therefore, it is only allowed to exclude the influence of the boundary conditions on the container from the momentum balance when considering a control box inside a much larger container. The momentum inside the control box depends on the shape of the control box. The momentum balance of the control box reads

$$F\delta(t-t_0) - \rho\int_{-Y}^{Y}[\Phi(-X, y, t) - \Phi(X, y, t)]\delta(t-t_0)\,dy$$
$$-\rho\int_{-Y}^{Y}\left\{\frac{\partial}{\partial t}[\Phi(-X, y, t) - \Phi(X, y, t)]\right\}H(t-t_0)\,dy$$
$$= \rho\int_{-X}^{X}\int_{-Y}^{Y}\frac{\partial\Phi}{\partial x}\delta(t - t_0)\,dx\,dy$$
$$+\rho\int_{-Y}^{Y}\left[-\left(\frac{\partial\Phi}{\partial x}\right)^2_{x=-X} + \left(\frac{\partial\Phi}{\partial x}\right)^2_{x=X}\right]H(t - t_0)\,dy \qquad (8)$$

Only the terms with $\delta(t - t_0)$ prove to yield a nonzero contribution. For simplicity, the volume integral of the rate of change of momentum is rewritten by partial integration, it yields a contribution at the singularity and one at the boundary, the result is

$$\rho\int_{-X}^{X}\int_{-Y}^{Y}\frac{\partial\Phi}{\partial x}\,dx\,dy = 2\pi a^2\rho U\delta(t - t_0)$$
$$-\rho\int_{-Y}^{Y}[\Phi(-X, y) - \Phi(X, y)]\,dy\delta(t - t_0) \qquad (9)$$

Of course the use of (9) reduces the balance (8) to an identity as the last terms on both sides of (8) vanish. The interesting thing is the momentum content of the control box after the time that the force $F\delta(t - t_0)$ has been working, it is found by the evaluation of (9) using (7)

$$MH(t - t_0) = F[1 - (2/\pi)\arctan(Y/X)]H(t - t_0) \qquad (10)$$

This means that the momentum content of the control box depends on its proportions. In a control box with $(Y/X) >> 1$, the force generates no net momentum, the momentum $\pi a^2\rho U$ of the circular cylinder of radius "a" moving with the dipole is balanced by a counterflow with momentum of equal magnitude in the fluid. In a square box with $Y = X$ (a circular cross section yields the same result), the momentum is half the impulse. It is the momentum $\pi a^2\rho U$ of the circular cylinder of radius a moving with the dipole. A box with $(Y/X) << 1$ contains the entire impulse F as momentum.

It should be noted that these results only hold on a small control box inside a large container. For a control box the size of the container the results must be adapted to take the boundary conditions into account. The boundary condition on a free atmosphere must be $p = 0$. Such a condition would require the addition of a homogeneous solution to the pressure field annihilating the unsteady pressure pulse on the walls at the onset of the motion. This additional field then imparts the total impulse as momentum to the whole of the fluid mass.

Momentum in the Two-Dimensional Field of a Moving Vortex Couple

The momentum, inside a control box, of a moving vortex couple with moment $2b\Gamma$, moving at a velocity $U = \Gamma/4\pi b$

(Fig. 3b) is found by integrating the momentum over the cross section of the control box:

$$M = \rho \int_{-X}^{X} \int_{-Y}^{Y} \frac{\partial \Phi}{\partial x} \, dx \, dy \tag{11}$$

Because the potential of a vortex couple has a jump Γ between the vortices, the integration with respect to x yields

$$M = \rho \int_{-Y}^{Y} [\Phi(X, y) - \Phi(-X, y)] \, dy + \rho(\Phi_L - \Phi_R)2b$$
$$(\Phi_L - \Phi_R) = \Gamma \tag{12}$$

Now letting X approach infinity before Y, the result of the momentum integral is

$$M = 2b\Gamma \tag{13}$$

But this is not a unique result because the integration was restricted to a control volume with $(Y/X) << 1$. Such a control box contains the whole of the impulse of the vortex couple as momentum.

A dipole with impulse equal to that of the vortex pair, and moving with the same velocity as the pair, would have moving with it a circular body of fluid of radius $2b$. The vortex pair has moving with it a body of fluid with oval cross section, the contour is described by

$$y = b \, \ell n \left[\frac{x^2 + (y - b)^2}{x^2 + (y + b)^2} \right] \tag{14}$$

The extreme dimensions of the oval are $x = \pm b\sqrt{3}$ and $y = \pm 2.0875b$, its area would be $3.6276\pi b^2$. The area of the oval is 0.9069 times the area $4\pi b^2$ of the circle associated with the equivalent concentrated dipole.

Considering the vortex couple to represent an element of the trailing vortex system of a lifting wing, it is correct to say that the moment $2\Gamma b$ is proportional (factor ρ) to the impulse (the element of lift force) that generated the vortex couple. It is also correct to state that it is the momentum inside the control box of Fig. 1b with $(Y/X) << 1$. The net momentum inside a control box with $(Y/X) >> 1$ is ZERO; the more so when we put a rigid surface with a boundary condition $\partial\Phi/\partial n = 0$ at some X. In a following section the drag aspect of a lifting wing is considered. The rigid boundary (the Earth's surface) for the lift component leaves the opportunity open for the horizontal (drag) component to have a different boundary condition, i.e., $p = 0$ on the end planes.

Two-Dimensional Flat Plate Moving Normal to Its Plane

A flat plate, width $2a$, moving normal to its plane with velocity U is not accompanied by a body of fluid moving with it. Its vortex distribution and its impulse are

$$\gamma(y) = \frac{-\Gamma y}{a\sqrt{a^2 - y^2}}, \qquad P = \frac{\pi}{4} 2aU\Gamma \tag{15}$$

No part of the impulsive force, required to set the plate into motion, is found back as momentum of a defined body of fluid. The infinite strip defined by each elementary vortex pair contains the entire momentum of that pair, but it also contains negative momentum of the pairs situated relatively more inward.

The plate has strong suction forces at the edges. The vortex distribution that kinematically represents the fluid motion cannot move in that flat shape when the supporting plate is removed. The edge vorticity would move away with the average of the infinite velocity next to the edge and the velocity of the plate. The flat sheet of bound vorticity left free will spirally concentrate the vorticity of its symmetric halves such that the impulse of the system is conserved.

It is interesting to consider a distribution of $2N$ discrete free vortices as a model to describe the fluid motion due to the moving flat plate. It yields a distribution of vortices of alternating sign. The average approaches the flat plate distribution, the amplitude of the variations around the average increases with increasing number $2N$.

Constant Forces Moving Steadily in an Atmosphere at Rest

Moving Actuator Surface

Consider a two-dimensional actuator surface A moving to the left with a velocity U and sustaining a pressure jump $p_R - p_L = \Delta p$, as shown in Fig. 4a. The thrust force (to the left) $T = \Delta pA$ is generated by a force T_{air} on the air to the right.

An actuator surface acts as a force that is continuously generating vorticity in the regions where $\nabla \times F \neq 0$. It so forms a slipstream or jet with increased total head. At the boundary between the jet and the external flowfield two vortex sheets are formed, they move with a velocity that is the mean of the velocities at both sides. This motion of the vortex sheets in their plane is irrelevant for the kinematics of the flowfield. For the dynamics, i.e., the unsteady pressure, it is of prime importance as it generates the increase in total head. This increase in total head is the constant value of $\partial\Phi/\partial t$ between the sheets due to the motion of the vortices. The disturbance in the external flowfield caused by the moving actuator surface is described by the potential of the vortex sheets. This latter potential is represented by a continuous distribution of dipoles in the slipstream. The integral in x direction yields a source distribution on the actuator surface, represented by one concentrated source, and a sink at the point where the actuator started its motion. The source strength is

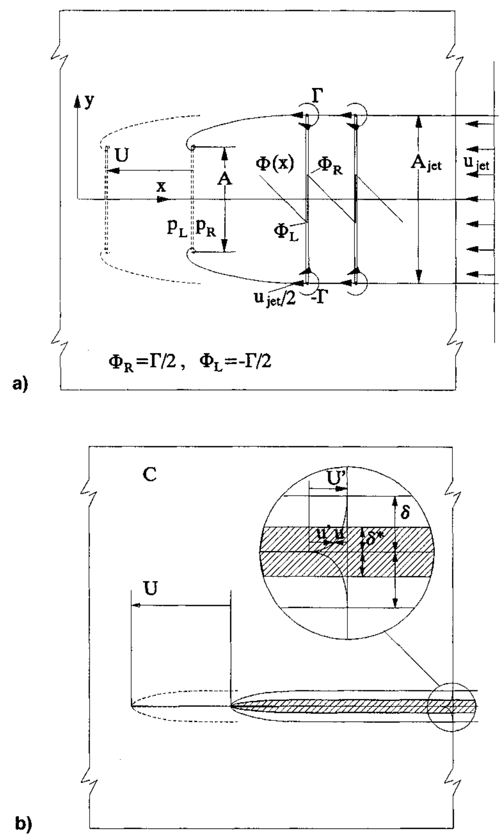

Fig. 4 a) Actuator surface moving in a nonviscous medium and b) a flat plate moving in a viscous medium.

$q = -A_{\rm jet}u_{\rm jet}$ (a negative $u_{\rm jet}$ yields a positive q). The source, moving to the left with the actuator, represents the blockage effect of the actuator surface. The sink moves with $u_{\rm jet}$ at the starting end of the jet. Note that the value of q is defined independent of T or U. It is the distinguished volume that per unit of time enters the control box between the vortex sheets, distinguished by a total head different from the main field.

The momentum balance (16) of a control volume $(2X, 2Y)$ reads

$$T_{\rm air} - \rho\int_{-Y}^{Y}\frac{\partial}{\partial t}\{\Phi[-X, y, \; -U(t-t_0), 0] - \Phi[X, y, \; -U(t-t_0), 0]\}\,{\rm d}y = \rho\int\int\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial x}\right){\rm d}x\,{\rm d}y + \rho\int_{-Y}^{Y}\left(\frac{\partial\Phi}{\partial x}\right)^2_{x=X}{\rm d}y \quad (16)$$

Note that U is used as a positive number, but $u = \partial\Phi/\partial x$ (in the slipstream $u_{\rm jet} = \partial\Phi/\partial x$) may be positive or negative. The mass $\rho u_{\rm jet}A_{\rm jet}$ that leaves the control volume per second with the momentum $\rho u^2_{\rm jet}A_{\rm jet}$ is supposed to be replenished without momentum. The terms in (16) are briefly discussed in the following text.

1) The integral of the unsteady pressure due to the moving source yields a net force on the control volume C. The sink, moving outside the volume C, yields an overall pressure variation and a zero force on C (it will yield a force on the control volume $C_{\rm rear}$). The unsteady pressure integral due to the moving source and sink yields

$$-\rho\int_{-Y}^{Y}\frac{\partial}{\partial t}[\Phi(-X, y) - \Phi(X, y)]\,{\rm d}y = -\rho\int_{-Y}^{Y}\left[(U + u_{\rm jet})\left(\frac{\partial\Phi}{\partial x}\right)_{x=-X} - (U - u_{\rm jet})\times\left(\frac{\partial\Phi}{\partial x}\right)_{x=X}\right]{\rm d}y = \rho Uq = -\rho Uu_{\rm jet}A_{\rm jet} \quad (17)$$

2) The area integral on the RHS of (16) is the increase of momentum inside the control volume. It consists of two parts, one is the increase of momentum in the slipstream due to the increase in length U per unit of time, the other is the increase in momentum content in the external flow (the blockage effect) due to the motion of the source over the distance U:

$$\rho\int\int\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial x}\right){\rm d}x\,{\rm d}y = \rho Uu_{\rm jet}A_{\rm jet} + \rho U\int_{\rm ext}\frac{\partial\Phi}{\partial x}\,{\rm d}y = \rho Uu_{\rm jet}A_{\rm jet} + \rho Uq = 0 \quad (18)$$

The last term may be explained as follows. When the source was immobile, half its mass would leave through the front area at $-X$, the other half through the area at $+X$. The momentum content of the control box would not vary. By the motion of the source the momentum content of the box decreases by $qU/2$ at left and increases by that amount at the right. The result is a variation with qU/s. It follows from (17) and (18) that this increase in momentum in the external flow due to the moving source is balanced by the pressure forces on the fore and aft faces due to the motion of this same source. The net momentum in a cross section of the atmosphere remains zero.

3) The second integral on the RHS of (16) is the transport of momentum in the jet through the boundary of the control box:

$$\rho\int_{-Y}^{+Y}\left(\frac{\partial\Phi}{\partial x}\right)^2_{x=X}{\rm d}y = \rho u^2_{\rm jet}A_{\rm jet} \quad (19)$$

The momentum balance that results when combining (17–19) formally is

$$T_{\rm air} + \rho qU = \rho u_{\rm jet}A_{\rm jet}(U + u_{\rm jet}) + \rho qU \quad (20)$$

leading to the well-known expression for the force on the air

$$T_{\rm air} = \rho u_{\rm jet}A_{\rm jet}(U + u_{\rm jet}) \quad (21)$$

An interesting aspect of the momentum balance (21) is the occurrence of the term with $u^2_{\rm jet}$. When the direction of the force is reversed, but not the direction of motion of the actuator surface, all terms except the one featuring $u^2_{\rm jet}$ change sign. This implies that a positive slipstream of cross section $A_{\rm jet}$ is generated by a larger force than a negative slipstream. The difference is caused that slipstreams with cross section $A_{\rm jet}$ and a positive or a negative velocity $u_{\rm jet}$ are generated by actuator surfaces of different size and pressure jump. The total force on the actuator surface follows from the momentum balance (20). Leaving out the contributions referring to the flow outside the slipstream (the unsteady pressure balances the outside momentum), we retain (21). This implies that the ratio of the magnitudes of forces on the positive and on the negative working actuator surfaces (with equal $|u_{\rm jet}|$ and $A_{\rm jet}$) is given by

$$\frac{F_+}{F_-} = \frac{U + |u_{\rm jet}|}{U - |u_{\rm jet}|} \quad (22)$$

The area ratio and the pressure loading cannot be found from this unsteady consideration, as there are no streamlines to refer to for the formulation of the continuity of mass. From a steady-state formulation (with reference axes moving with the actuator), it follows that

$$\frac{A}{A_{\rm jet}} = \frac{U + u_{\rm jet}}{U + (u_{\rm jet}/2)}, \qquad \Delta p = \rho u_{\rm jet}\left(U + \frac{u_{\rm jet}}{2}\right) \quad (23)$$
$$F = \Delta pA = \rho u_{\rm jet}A_{\rm jet}(U + u_{\rm jet})$$

Note that the pressure loading is different from what one would expect from the increase in total head in the slipstream, i.e., $\Delta p \neq \rho u^2_{\rm jet}/2$. As a loose rationalization in this unsteady approach one might say that the force (21) is built up from the increase in total head times the cross section plus the force on the moving source or sink. This latter force consists of the jump in $\rho\partial\Phi/\partial t$ in the points where the actuator surface (bearing the source as a source layer) passes through. This jump is quantified as

$$(p_R - p_L)_{\rm source}A = -\rho\int_A\Delta\left(\frac{\partial\Phi}{\partial t}\right){\rm d}y = \rho\int_A\Delta\left(\frac{\partial\Phi}{\partial x}\right)\frac{\partial x_0}{\partial t}\,{\rm d}y = -\rho qU = \rho u_{\rm jet}A_{\rm jet}U \quad (24)$$

It is interesting to consider the momentum balance of the control volume $C_{\rm rear}$. There is no external force working on it. The pressure integrals due to the moving sink now yield a net contribution, whereas the source, moving inside C, yields an overall pressure variation and no net force on $C_{\rm rear}$. The pressure integrals yield

$$\int_{\rm Trefftz} -\rho\frac{\partial\Phi}{\partial t}\,{\rm d}y - \int_{\rm aft} -\rho\frac{\partial\Phi}{\partial t}\,{\rm d}y = -\rho(U - u_{\rm jet})\frac{q}{2} + \rho(U + u_{\rm jet})\frac{q}{2} \quad (25)$$

The rate of change of the momentum content of C_{rear} has two contributions that, analogous to (18), annihilate each other. The jet increases in length with u_{jet} per unit time and the motion of the sink affects the momentum of the field outside the jet.

The momentum transport in the Trefftz-plane yields the contribution

$$-\int_{Trefftz} \rho u^2 \, dy = -\rho u_{jet}^2 A_{jet} \tag{26}$$

The momentum balance of C_{rear} leads to the identity

$$\rho q u_{jet} = -\rho u_{jet}^2 A_{jet} \tag{27}$$

Merging the momentum balances of C and C_{rear} to obtain the balance of C_{total} the pressure integral over the Trefftz-plane drops out. It proves that, as long as no boundary conditions are applied, the force T_{air} is balanced half by the pressures on the fore and half by the pressure over the aft area:

$$T_{air} + \int_{fore} p \, dy - \int_{aft} p \, dy = 0$$
$$\int_{fore} p \, dy = \frac{\rho q}{2}(U + u_{jet}) \tag{28}$$
$$-\int_{aft} p \, dy = \frac{\rho q}{2}(U + u_{jet})$$

Now when the boundary condition is applied that requires the pressure integrals to vanish in a free atmosphere, an accelerating pressure gradient must be imposed yielding the negative of the pressure integrals on the end faces of C_{total}. This pressure gradient accelerates the air inside C_{total} such that the rate of increase of the momentum of C_{total} equals T_{air}. As this counter-momentum is imparted to the control box C_{total}, the associated pressure forces on the faces of C are smaller than T_{air} by an order $\mathcal{O}(l_1/l_2)$. In a realistic atmosphere $l_2 >> l_1$ and in a control box C, spanning a slice l_1 of the atmosphere the net momentum remains zero.

The energy balance of the actuator surface in the control box C reads

$$-T_{air}U + T_{air}[U + (u_{jet}/2)] = \rho(u_{jet}^2/2)(U + u_{jet})A_{jet} \tag{29}$$

In words it says that the difference between the external work delivered by T and the power put into the actuator is found back as kinetic energy in the flow. Finally, this kinetic energy will be dissipated into heat.

Two-Dimensional Flat Plate Moving in a Viscous Fluid

Consider a control volume C with sides $2X$, $2Y$, where $Y >> X$, as shown in Fig. 4b. The shape of this control volume implies that the applied force F will be mainly balanced by unsteady pressure forces on the control boundary. No net momentum will be formed in the box, there is only transport through the boundary. The momentum balance involves two aspects. The primary aspect is that the viscous force entrains fluid, it delivers negative momentum. The secondary aspect is that the entrained fluid acts as a half-infinite body intruding into the control volume. The $\partial\Phi/\partial t$ term generates a high pressure at the left area and a low pressure at the right, it represents the blockage effect of the boundary layers.[4] The net unsteady-pressure force, having a constant magnitude, is associated with a steady increase in momentum in the field outside the boundary layers, complementary to the negative momentum generated in the boundary layers. The momentum balance is

$$F_{air} + \int_{left} p \, dy - \int_{right} p \, dy = \rho \int\int \frac{\partial}{\partial t} u \, dx \, dy + \rho \int_{-\infty}^{+\infty} u_{right}^2 \, dy \tag{30}$$

Leading finally to $F_{air} = -2\rho U^2\theta$, where θ is the momentum thickness of one boundary layer. Note that the velocity u is not the velocity u' that is usually used in the definition of the momentum thickness, here, $u = u' - U$, as shown in Fig. 4b. Substitution of this relation in the integrals over the boundary layers leads to the following expressions in terms of δ^* and θ:

$$\int_{-\delta}^{+\delta} u \, dy = -\int_{-\delta}^{+\delta} (U - u') \, dy = -2U\delta^*$$
$$\int_{-\infty}^{+\infty} u^2 \, dy = \int_{-\delta}^{+\delta} u^2 \, dy = \int_{-\delta}^{+\delta} (U - u')^2 \, dy \tag{31}$$
$$= 2U^2\delta^* - 2U^2\theta$$

Before the terms of (30) are discussed in detail the source strength is quantified in terms of δ^*. The force F_{air} of the plate on the fluid generates momentum in thin boundary layers feeding a wake increasing in length with U per unit of time. The plate entrains fluid to an amount $2U\delta^*$ per unit of time, where δ^* is the displacement thickness of one boundary layer. This entrained fluid acts on the surroundings as a long body being pushed into the fluid. Its external potential field is simulated by a two-dimensional source of strength $q = 2U\delta^*$ per unit width, moving with the plate and a distribution of sinks of strength u dy, moving with velocity $u(y)$, at the location where the plate started its motion. In fact, the source and the sink distribution are the result of the integral of a growing distribution of dipoles, in $-x$ direction, representing the distribution of discontinuities in the potential of the vorticity layers of opposite sign forming the boundary layers. The motion of the source generates an unsteady pressure field as well as an increase of the momentum content of the volume outside the boundary layers. The motion of the sink has an analogous effect in C_{rear}.

The pressure integrals on the left-hand side (LHS) of (30) are evaluated using Bernoulli's unsteady law where the dynamic head, being quadratic in the small velocities, is neglected. The pressure force on the left area due to the moving source q is

$$\int_{left} p \, dy = \int_{-\infty}^{+\infty} -\rho \frac{\partial\Phi}{\partial t} \, dy$$
$$= -\int_{3\pi/2}^{\pi/2} \frac{\rho q U}{2\pi r} \cos\phi \frac{r \, d\phi}{\cos\phi} = \frac{\rho q U}{2} \tag{32}$$

For the effect of the sink consider the final boundary layer as the result of elementary actuators yielding layered slipstream elements of width dy with velocity $u(y)$. The unsteady pressure due to a moving sink of intensity u dy, in the end of the far right boundary layer, moving with velocity u yields a force of magnitude $-\rho u^2 \, dy/2$ on the left area. Integrated over the boundary layers the force on the left area due to the elementary sinks amounts to

$$\int_{left} p_{sink} \, dy = -\frac{\rho}{2} \int_{Trefftz} u^2 \, dy \tag{33}$$

The pressure forces on the right area are computed in an analogous way, there the source yields a result equal to (32), the sink yields a result opposite to (33). The total pressure force on the control volume thus becomes

$$\int_{\text{left}} p\,dy - \int_{\text{right}} p\,dy = 2\rho U^2\delta^*$$

$$\begin{aligned} \int_{\text{left}} p\,dy &= \frac{1}{2}\left(\rho Uq - \int_{\text{Trefftz}} \rho u^2\,dy\right) = \rho U^2\theta \\ -\int_{\text{right}} p\,dy &= \frac{1}{2}\left(\rho Uq + \int_{\text{Trefftz}} \rho u^2\,dy\right) \\ &= 2\rho U^2\delta^* - \rho U^2\theta \end{aligned} \tag{34}$$

A comment on the pressure inside the boundary layer is appropriate. The pressure in the boundary layer is implicitly taken equal to the pressure in the adjacent nonviscous flow. Schematically modeling the boundary layer as two moving vortex layers at its edges, as in Fig. 4a, the pressure between the layers is to be found from Bernoulli's unsteady equation involving $\rho u^2/2$ and $\rho\partial\Phi/\partial t$. The unsteady contribution to the pressure is $-\rho\partial\Phi/\partial t$, quantified as

$$-\rho\frac{\partial\Phi}{\partial t} = -\rho\frac{\partial\Phi}{\partial x_0}\frac{\partial x_0}{\partial t} = \rho\frac{\partial\Phi}{\partial x}\left(-\frac{|u|}{2}\right) = \rho\frac{u^2}{2} \tag{35}$$

The total head in the free atmosphere is $p_0 = p$. Define a new total head in the schematic wake

$$p_0' = p_0 - \rho\left(\frac{\partial\Phi}{\partial t}\right)_{\text{wake}} = p_{\text{wake}} + \rho\frac{u_{\text{wake}}^2}{2} \tag{36}$$

This new total head p_0' is higher than the total head p_0 in the surroundings (see also Ref. 5), the static pressure p_{wake} is in equilibrium with the surrounding pressure p. This implies that in the static pressure p_{wake} the contribution of $\rho u^2/2$ is only implicitly present, whereas in the momentum balance (30) a term involving u^2 appears explicitly.

The increase of the momentum content of the volume consists of two parts, a part inside the boundary layers and a part in the potential field outside the boundary layers:

$$\rho\int\int\frac{\partial}{\partial t}u\,dx\,dy = \rho U\int_{-\delta}^{+\delta}u\,dy + \rho\frac{\partial}{\partial t}\int_{\text{ext}}\int\frac{\partial\Phi}{\partial x}dx\,dy \tag{37}$$

The part inside the boundary layers is rewritten as

$$\rho U\int_{-\delta}^{+\delta}u\,dy = -2\rho U^2\delta^* \tag{38}$$

The part exterior to the boundary layers, indicated by "ext," is due to the motion of the source, it comes forward as follows. A cross-sectional slice of thickness dx to the left of the source has a momentum content

$$dM = dx\int_{\text{left}}\rho\frac{\partial\Phi}{\partial x}dy = \frac{\rho q}{2\pi}dx\int_{\text{left}}\frac{\cos\theta}{r}dy = -\frac{\rho q\,dx}{2} \tag{39}$$

Per unit of time the source moves over the distance U to the left so that at the left of the source the momentum content decreases with $-\rho qU/2$, i.e., it increases with $\rho qU/2$, and at the right it increases with $+\rho qU/2$. The contributions together yield the momentum increase ρqU:

$$\frac{\partial}{\partial t}\int_{\text{ext}}\int\rho u\,dx\,dy = \rho qU \tag{40}$$

The first term on the RHS of (30), quantified in (38) and (40), yields a zero contribution. The choice of the shape of the control box already implied that there is no net momentum generated by the force F inside C.

The momentum transport through the boundary has only a significant contribution inside the boundary layers. It is, rewritten in terms of δ^* and θ:

$$\int_{\text{right}} u^2\,dy = \int_{-\delta}^{+\delta} u^2\,dy = 2U^2(\delta^* - \theta) \tag{41}$$

Writing F_{pl} and F_{pr} for the left- and right-pressure integrals, the momentum balance becomes

$$\begin{aligned} F_{\text{air}} + F_{\text{pl}} + F_{\text{pr}} &= \rho\int_{-\infty}^{+\infty}u^2\,dy \\ F_{\text{air}} + 2\rho U^2\delta^* &= -2\rho U^2\theta + 2\rho U^2\delta^* \\ \text{or}\quad F_{\text{air}} &= -2\rho U^2\theta \end{aligned} \tag{42}$$

The conclusion is, taking into account the interpretation of the terms (38) and (40), that the pressure forces on the control surface due to the blockage effect generate an increase of momentum outside the boundary layers. The friction force F generates the momentum in the viscous wake, including the transport term. The net momentum in a slice dx of the atmosphere remains zero.

At the boundary where the wake intrudes into the control volume, momentum is transported from one part of the atmosphere to another.

When the control box C_{rear} is considered, extended to a location far behind the starting point of the plate where only negligible velocities are present, the momentum balance becomes

$$\begin{aligned} \int_{\text{Trefftz}} p\,dy - \int_{\text{aft}} p\,dy &= -\rho\int_{\text{Trefftz}} u^2\,dy \\ \int_{\text{Trefftz}} p\,dy &= -\frac{1}{2}\left(\rho Uq + \int_{\text{Trefftz}}\rho u^2\,dy\right) \\ &= -2\rho U^2\delta^* - \rho U^2\theta \\ -\int_{\text{aft}} p\,dy &= \frac{1}{2}\left(\rho Uq - \int_{\text{Trefftz}}\rho u^2\,dy\right) = \rho U^2\theta \end{aligned} \tag{43}$$

The momentum balance of the total atmosphere gets the form

$$\begin{aligned} F_{\text{air}} + \int_{\text{fore}} p\,dy - \int_{\text{aft}} p\,dy &= 0 \\ \int_{\text{fore}} p\,dy = -\int_{\text{aft}} p\,dy &= \rho U^2\theta \end{aligned} \tag{44}$$

It is clear that no momentum is present when the end faces can support the pressure. Or in other words, the blockage effect guarantees that the net momentum in any slice of the atmosphere remains zero. When the boundaries are not supported (they may be thought of as being connected in a loop) the force must deliver net momentum to the atmosphere. A net acceleration is achieved by pressures on the boundary areas of C_{total}. Addition of these pressures and the associated momentum then results in the situation that the (negative) force F_{air} brings a net momentum $-2U^2\theta$ per unit of time into the atmosphere. This momentum is explicitly present in the boundary layers, where it formally remains while diffusing into the surroundings. Nevertheless, the net momentum in a slice of atmosphere remains practically zero. To see this consider the atmosphere inside a box C_{total} of length $(l_1 + l_2)$ much larger than the distance between source and sink, the length of the wake. In that case the blockage effect, existing

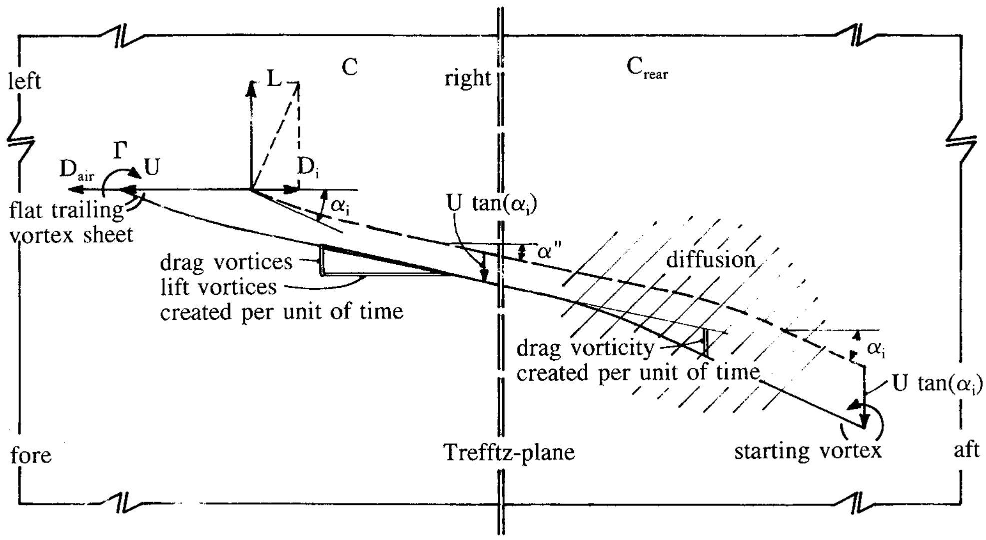

Fig. 5 Lifting wing and its trailing vortex system.

between source and sink, is balanced by momentum of a much larger mass of air involving much smaller velocities than the blockage effect.

Induced Drag of a Three-Dimensional Lifting Wing Moving in the Atmosphere

Consider a lifting wing moving in the control volume C. The trailing vortex system extends through the right face, the Trefftz-plane, into the rear control volume C_{rear} where it is closed by the starting vortex, as sketched in Fig. 5.

The overall momentum balance in the x direction, covering C and C_{rear}, states that the drag force D_{air} is balanced by unsteady pressure forces on the fore and aft faces. The net momentum in any cross section remains zero and the free atmosphere is accelerated by the pressure gradient that annihilates the pressure integrals on the fore and aft areas.

It is customary to consider only the control volume C, bounded by the Trefftz-plane. Formally, the phenomena taking place in C_{rear} do not affect the momentum balance of C, but in connection with the overall momentum balance the unsteady-pressure effects generated in C_{rear} must be registered.

The x component of the momentum balance of C is (using $D_{air} = -D_{wing}$):

$$D_{air} + \int\int (p_{left} - p_{right})\, dy\, dz$$
$$= \rho \int\int U \frac{\partial\Phi}{\partial x}\, dy\, dz + \rho \int_{right}\int \left(\frac{\partial\Phi}{\partial x}\right)^2 dy\, dz \qquad (45)$$

The terms of (45) are discussed later in the following text:

1) The contributions of the pressure forces, the second term on the LHS of (45), are different from those generated by the actuator surface and the boundary layer in a viscous flow. The drag of the lifting wing is not associated with a jet or wake of which the total head is modified, therefore, there is no direct association with an unsteady source potential. Nevertheless, it proves that an equivalent to the blockage effect can be identified in the pressure field. The wing steadily generates connected ring vortices with impulse components proportional to L and D. These ring vortices are not connected to form a jet, they form a slanted vortex sheet that rolls up behind the wing. In the process of rolling up, the lift component is supposed to retain its impulse. The impulse of the drag component is modified in the pressure field behind the moving wing. The rolled-up system of vortices in the Trefftz-plane is slanted at an angle α'' in agreement with its self-induced downward motion. The horizontal impulse of the rolled-up system is smaller than the horizontal impulse of the flat sheet behind the wing, slanted at an angle α_i. It is implicitly understood that the rolled-up vortices have axial momentum in their core regions.[8-11] The unsteady pressure field in the Trefftz-plane is due to the increase in length of the trailing vortex system, to the downward motion in the Trefftz-plane, and to the phenomena in C_{rear}. To describe the processes the growing vortex system is discussed in three parts. Part one consists of the slanted parallel vortices moving downward with $U \tan \alpha''$, part two takes account of the formation of new vorticity inside C, and part three considers the effects of the phenomena in C_{rear}.

Part one, the downward motion of the drag component has no net effect on the pressure integral in the Trefftz-plane, the motion of the lift component yields a net overpressure. For the evaluation of the unsteady pressure integral due to the downward motion of the lift component, it is not necessary to define the shape of the integration area in the Trefftz-plane. The presence of the ground guarantees that the net vertical momentum vanishes. The presence of the factor (dz_0/dt) in the integral (46) changes the sign of the part of $(\partial\Phi/\partial z)$ that is due to the image in the ground. The result of the integral therefore equals the total impulse of the vortex pair in a free atmosphere multiplied by $U \tan(\alpha'')$. The distance between the rolled-up vortices of an elliptically loaded wing of span b is $\pi b/4$. The unsteady pressure integral due to the downward motion of the lift vortices has the form:

$$-\int_{-\infty}^{\infty} dy \int_0^{\infty} \left(-\rho \frac{\partial\Phi}{\partial t}\right) dz = \rho \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \frac{\partial\Phi}{\partial z_0}\frac{dz_0}{dt}\, dz$$
$$= \rho \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial\Phi_{free\,space}}{\partial z}\, U \tan(\alpha'')\, dy\, dz$$
$$= -\frac{\pi}{4}\rho b \Gamma U \tan(\alpha'') \qquad (46)$$

This means that the downward motion of the vortex pair generates an overpressure.

The second part of the decomposition consists of the rate of increase in length of the vortex system, associated with the formation of lift vortices of length U per unit of time and vertical drag vortex rings of height $U \tan(\alpha'')$. The potential of the formed lift vortices is described by the generation per unit of time of a vertical dipole moment $L/\rho = (\pi/4)bU\Gamma$. The associated unsteady pressure integral over a y-z plane at a remote x station vanishes.

The horizontal dipole moment being formed per unit of time is $(\pi b/4)\Gamma U \tan(\alpha'')$. Its impulse is smaller in magnitude than the drag force per unit mass, it has been modified in the pressure field in the wake. The dipole is equivalent to the time-derivative of the moving source modeling the potential of the growing row of vortices bounding the jet in the preceding section. It represents the blockage effect of the lifting wing yielding a high pressure ahead of and a low pressure behind the latter. The contribution of the associated unsteady pressure integral in the Trefftz-plane is

$$\rho \int_{\text{Trefftz}} \int \Phi_{\text{dipole}} \, dy \, dz$$

$$= \rho \int_{\text{Trefftz}} \int \frac{\pi}{4} bU\Gamma \tan(\alpha'') \frac{(x - x_0)}{4\pi R^3} \, dy \, dz$$

$$= \rho \frac{\pi}{8} b\Gamma U \tan(\alpha'') \tag{47}$$

The contributions of parts one (47), and two (48), result in a net overpressure yielding a force:

$$\rho \int_{\text{right}} \int \frac{\partial \Phi}{\partial t} \, dy \, dz = -\rho \frac{\pi}{8} b\Gamma U \tan(\alpha'') \tag{48}$$

As previously mentioned, the formation of the drag dipole potential generates an unsteady pressure on the left area of the control box, the high pressure on the left area yields a contribution equal to that of the low pressure in part three of the right integral (47). The contribution of the pressure integral over the left area is

$$\int_{\text{left}} p \, dy \, dz = \rho \frac{\pi}{8} b\Gamma U \tan(\alpha'') \tag{49}$$

The contributions (48) and (49) cancel out on the control volume C, they represent an overpressure on both end faces of C. One might say that the blockage effect is masked by the overpressure due to the lift system.

Part three concerns the effects of the phenomena in the rear control area. At the start of the motion of the wing the vortex system was slanted downward at an angle α_i. The vertical velocity of the starting vortex was, roughly speaking, quantified as $U \tan(\alpha_i)$. In the ensuing extension and rolling-up, a quasi-two-dimensional field developed, the slanting angle of which got modified to become α'' under the influence of the dynamic pressure field. The axial pressure gradients gave rise to rearward flow in the vortex cores,[9,10] at the wing-end of the rolled-up vortex system. The other end of the trailing vortex system, where diffusion dominates, gradually developed into an ordinary wake with an axial velocity defect (Ref. 9, p. 654). To quantify the effect in the model it is necessary to refer to the sections on the actuator surface and the viscous boundary layer. In these sections the end of the slipstream and the wake were modeled as moving sinks. The motion of these sinks contributed to the unsteady pressure on the end faces of C_{total} to such an extent that it made up for the difference in magnitude between the applied force and the impulse of the vortices generated inside C per unit of time. The description of the behavior of the diffusing trailing vortex system in C_{rear} is kept purposedly vague and general. The effect of the phenomena must be such that the associated unsteady pressure integrals, referred to as $P_{\text{rear}}/2$, over the fore and over the aft area of C_{total}, make up for the difference in magnitude between D_{air} and the x impulse, i.e., $-\rho(\pi/4)bU\Gamma \tan(\alpha'')$, of the vortices generated inside C per unit of time. The associated pressure integrals on the left and right faces of C cancel out. Formally, the contribution $-P_{\text{rear}}/2$ must be added to the pressure integral over the Trefftz-plane of C and $+P_{\text{rear}}/2$ must be added to the left pressure integral of C.

The pressure integral over the Trefftz-plane, in a steady or in an unsteady treatment, thus gets the form:

$$-\int_{\text{right}} p \, dy \, dz = \int_{\text{right}} \frac{\rho}{2}(u^2 + v^2 + w^2) \, dy \, dz$$

$$- \frac{\pi}{8} \rho Ub\Gamma \tan(\alpha'') - P_{\text{rear}}/2$$

$$= \int_{\text{right}} \frac{\rho}{2}(u^2 + v^2 + w^2) \, dy \, dz + D_{\text{air}}/2 \tag{50}$$

Another way of looking at the unsteady pressure integral in the Trefftz-plane is to model the increase in length as just a motion of the system to the left. The partial time derivative of Φ gets the form $\partial\Phi/\partial t = -U\partial\Phi/\partial x_0 = U\partial\Phi/\partial x$. The pressure integral over the right area (the Trefftz-plane) would yield:

$$-\int_{\text{right}} p \, dy \, dz$$

$$= -\int_{\text{right}} \left[p_0 - \frac{\rho}{2}(u^2 + v^2 + w^2) - \rho \frac{\partial \Phi}{\partial t} \right] dy \, dz$$

$$= \int_{\text{right}} \frac{\rho}{2}(u^2 + v^2 + w^2) \, dy \, dz + \rho \int_{\text{right}} \frac{\partial \Phi}{\partial x} U \, dy \, dz \tag{51}$$

The interpretation of $\partial\Phi/\partial t$ as $\partial\Phi/\partial t = -U\partial\Phi/\partial x_0 = U\partial\Phi/\partial x$ (referred to as the steady interpretation) does not take into account the stretching of the vortex system. In the unsteady description the front end of the system moves to the left, the rear end remains at its x station while only moving down. This means that in a steady interpretation the growth of the vortex system takes place in the volume C_{rear}, then it yields an unsteady high-pressure contribution equal to $|D_{\text{air}}/2|$ on both the left and the right end face of a steady volume C. In the unsteady treatment the growth of the vortex system takes place in C.

2) The first integral at the RHS of (45) is the increase of momentum in the control volume. The momentum increase is U times the momentum in a section $x = \text{const}$:

$$\int \int \rho U \frac{\partial \Phi}{\partial x} \, dy \, dz = 0 \tag{52}$$

This expression includes the momentum in the vortex cores. Continuity of the mass flow implies that the integral of u over the full area vanishes.

3) The momentum transport through the downstream control area yields the identically positive contribution

$$\int_{\text{right}} \rho u^2 \, dy \, dz \tag{53}$$

When this integral is based on such u distribution as the axial flow in the vortex cores the main contribution will come from the regions with high $|u|$. [As shown in (4) and Fig. 1, it does not matter whether the peaks concern positive or negative velocities, the square guarantees a positive contribution in the Trefftz-plane.]

Combining the terms (49) + $P_{\text{rear}}/2$, (50), (52), and (53), the momentum balance in the x direction yields the well-known expression for the drag

$$D_{\text{air}} = -\int_{\text{right}} \left[\frac{\rho}{2}(u^2 + v^2 + w^2) - \rho u^2 \right] dy \, dz \tag{54}$$

Energy

Multiplying both sides of (54) by $-U$, one obtains (55), a relation with the dimension of power

$$-UD_{\text{air}} = +\rho U \int_{\text{right}} \int \left[\frac{(u^2 + v^2 + w^2)}{2} - u^2 \right] dy\, dz \tag{55}$$

The interpretation of this form becomes clear when considering the energy balance of the control volume in a free atmosphere where the boundary condition on the area ahead of the wing puts the net pressure force on that area to zero. The pressure integral on the Trefftz-plane then gets the form (51). The energy balance becomes

$$\begin{aligned} -UD_{\text{air}} &+ \int_{\text{right}} \rho U u^2 \, dy\, dz \\ &+ \rho \int_{\text{right}} u \frac{(u^2 + v^2 + w^2)}{2} dy\, dz \\ &= \rho U \int \frac{(u^2 + v^2 + w^2)}{2} dy\, dz \\ &+ \rho \int_{\text{right}} u \frac{(u^2 + v^2 + w^2)}{2} dy\, dz \end{aligned} \tag{56}$$

The LHS contains the contributions of the forces that perform work, i.e., the drag, the unsteady pressure, and the dynamic pressure. The RHS lists the result, the increase in kinetic energy per unit time due to increase in length of the wake corrected for the transport through the boundary. It proves that the $\rho U u^2$ term in (56), and therefore also in (55), represents the work performed by the surroundings, i.e., the volume C_{rear} downstream of the control volume C. In the volume C_{rear} the kinetic energy is dissipated into heat. The pressures are in equilibrium. The increase in heat content per unit of time consists of the dissipation of the kinetic energy minus the work performed by the volume C_{rear} on the control volume C. The increase equals DU. The work DU is delivered by the external force towing the wing, in the steady model the work is of course performed by the pressure forces on the aft area (the fan in a wind tunnel).

Rear Control Volume

The momentum balance of the control volume C_{rear} contains only the pressure integrals and the momentum transport through the Trefftz-plane. In this part of the atmosphere the trailing vortices eventually approach the ground and curl forward.[10] This curling forward does not affect the horizontal impulse of the vortex system. The inclination of the system in the vertical plane is substituted by an inclination in the horizontal plane. The two vortices curl forward paired with their images in the ground at a vertical distance $(\pi/4)b$.

The processes at the end of the vortex system, described earlier, yield a contribution $P_{\text{rear}}/2$ on the Trefftz-plane and an equal contribution on the aft face of C_{rear}. The pressure integral over the Trefftz-plane is the negative of (50), the integral over the aft plane is the unsteady pressure (48) plus $P_{\text{rear}}/2$. The momentum balance then yields

$$\begin{aligned} &\int_{\text{Trefftz}} p \, dy\, dz - \int_{\text{aft}} p \, dy\, dz \\ &- \int_{\text{Trefftz}} \frac{\rho}{2} (u^2 + v^2 + w^2) \, dy\, dz \\ &+ \frac{\pi b}{4} \rho U \Gamma \tan(\alpha'') + P_{\text{rear}} = -\rho \int_{\text{Trefftz}} u^2 \, dy\, dz \end{aligned} \tag{57}$$

Merging the control volumes C and C_{rear} to form a control volume C_{total}, the Trefftz-plane drops out. The result [Eq. (58)] then is analogous to (57), only the integrals of dynamic head and ρu^2 have made place for D_{air} in agreement with (55):

$$D_{\text{air}} + \frac{\pi b}{4} \rho U \Gamma \tan(\alpha'') + P_{\text{rear}} = 0 \tag{58}$$

The fact remains that at intermediate distances behind the wing the increase per unit time of the kinetic energy in the wake is greater than the work performed by the wing. It is reassuring to know that not all that kinetic energy is diffusing into heat. Only the part of the kinetic energy that is proportional to the work performed by the wing is degenerated.

Pressure in the Trefftz-Plane

The pressure integral over the Trefftz-plane as an end face of the control volume C is represented in (50). This expression suggests that there is a low pressure on the Trefftz-plane yielding a net pressure force $(D_i/2 + \int \rho u^2)$. One may wonder, with Sears,[6,7] whether in some model situation the Trefftz plane experiences a high pressure. This high pressure should come from the integral of $\rho U(\partial\Phi/\partial x)$ as it occurs in (51). The quadratic terms definitely yield a low-pressure contribution. Sears does suggest that in some model situations (e.g., "flat sheet" wake), the unsteady pressure term in (51) may yield so large a high-pressure contribution that the overall effect becomes a pushing force on C. The reasoning is as follows: let the wake of axial vortices behind a lifting wing be slanted downward by an angle α'', which is a factor k times α_i, $\alpha'' = k\alpha_i$, where $D_{\text{air}} = -L \tan(\alpha_i)$. The linear integral of $\rho U(\partial\Phi/\partial x)$ in Sears' reasoning is related to the integral of the z momentum in the Trefftz-plane as

$$\begin{aligned} -\rho \int_{\text{right}} U \frac{\partial \Phi}{\partial x} dy\, dz &= -\rho \int_{\text{right}} U w'' \sin(\alpha'') \, dy\, dz \\ &\approx -\rho \int_{\text{right}} k U w'' \sin(\alpha_i) \, dy\, dz \\ &= -\rho k \tan \alpha_i L = -\rho k D \end{aligned} \tag{59}$$

This is the step that is discussed next. Two comments on the model and the integral are made. One concerns the rolling up of the vortex sheet and the consequence for the momentum integral, the other concerns the association of the momentum integral with lift.

1) The flat vortex sheet cannot exist as a free sheet because there would be an infinite suction force at the edge, and in reality it rolls up. For the computation of the induced velocities at the wing the flat sheet is useful and satisfactory in elementary lifting-line theory, for use in a momentum balance it is useless as it violates dynamic rules. The spanwise rolling-up of a two-dimensional sheet avoids the infinite pressure at the edge, the variation of the slope in x direction is associated with a redistribution of momentum in that direction.[8–11] Considering the sheet far downstream to be rolled up to form a vortex pair with vertical impulse L/ρ per unit length, as is usually done, it is not unreasonable to associate the vertical component of the vortex pair with an equivalent horizontal impulse. Sears suggests associating the integral of $\rho U(\partial\Phi/\partial x)$ with $\rho k D$. This association is based on the interpretation that the integral of $\rho U(\partial\Phi/\partial x)$ represents the impulse, whereas it is momentum. The net momentum integral over a cross section of the atmosphere yields zero. The relation of this integral to impulse is only consistent when the integration area in the Trefftz-plane is restricted to a vertical strip between concentrated vortices. For the flat sheet model such a strip would not have the width of the sheet, as pointed out before. A more realistic value of k will come forward in a following section.

2) In the process of rolling-up two aspects must be distinguished: 1) the impulse of the vertical components of the vortices is modified in the pressure field behind the wing and 2) the linear relation between the horizontal and the vertical momentum components of the vortices is broken by the generation of strong axial components in the vortex cores. The relation involving $\tan(\alpha_i)$ between the $\partial\Phi/\partial x$ integral and the w'' integral in (59) is only valid when the spirally wound core regions are excluded.

It must be stated that the dynamic pressure force on the entire Trefftz-plane balances the induced drag plus the momentum transport. Unsteady effects raise the pressure integral over the Trefftz-plane by an amount $|D_{\text{air}}/2|$. Even in a steady treatment, considering the lifting wing in a flow U, this contribution would be present. Then it is due to the unsteady phenomena taking place downstream of the Trefftz-plane.

The pressure distribution in the Trefftz-plane in a rolled-up model must be based on the model of the potential flow excluding the vortex cores. Consider a Trefftz-plane that is restricted to a vertical strip between the trailing vortices. The entire lift of the airplane is to be found as vertical momentum in this strip of the atmosphere. Due to the inclination of the wake the vertical momentum is related to the horizontal momentum excluding the core regions. The evaluation of the equivalent pressure integral must be based on the potential flow part.

Considering a flat sheet wake the factor k will have the value $k = 2$ so that $\alpha'' = 2\alpha_i$, and interpreting the $\rho U\partial\Phi/\partial x$ integral on the strip as to yield the impulse one would conclude to a net overpressure on the strip based on the relation $2\alpha_i L = 2D$. But, as mentioned in the section on the plate moving normal to its face, the momentum integral in that case does not yield the impulse. More important is that a realistic value of k strongly differs from $k = 2$. The downward velocity w'' of two concentrated vortices and the downwash w of an equivalent flat sheet are related as $w''/w = 4/\pi^2$, making $k = 4/\pi^2$ instead of $k = 2$.

The momentum balance of C_{total} states that the drag force is balanced by the low unsteady pressure on the aft area relative to the fore area. When a boundary condition $p = 0$ is explicitly applied on the end faces of the control volume, the external drag force results in momentum of the entire atmosphere.

A self-propelling aircraft in steady horizontal flight does not bring net momentum in the atmosphere, it does bring into the atmosphere a momentum distribution, and thus, it adds net energy.

Discussion and Conclusions

The kinetic energy in the wake region of a lifting wing must not solely be attributed to the work performed by the drag force, it is greater than that by the work performed by the surroundings. The heat content of the atmosphere is not increased by an amount as would follow from the dissipation of the kinetic energy, it is less than that. A part of the energy, to the amount of the ρUu^2-integral/s, is recuperated and transmitted forward.

Comparing the actuator surface with the viscous boundary layer it is noted that the power taken out by an actuator is only part of the work performed on the system from outside. The difference initially remains as kinetic energy in the field and is finally turned into heat. In the viscous boundary layer all the work put in from outside turns into heat.

The recovery mechanism that worked with the wing over the entire Trefftz-plane does also work with the actuator and the boundary layer as far as the flow outside the slipstream and the wake is concerned.

There is a remarkable difference in the treatment of the actuator surface exerting a forward force on the air and the induced drag of a wing, it concerns the pressure on the part of the control surface that is cutting through the wake. The actuator surface is associated with a positive unsteady contribution, due to the $-\rho\partial\Phi/\partial t$ term. This positive effect is annihilated by the dynamic head $\rho u^2/2$. The static pressure in the jet has the ambient value and so the jet cross section yields a zero contribution to the integrated pressure on the Trefftz-plane. Outside the jet the potential of the growing row of vortices is the potential of a forward-moving source. The unsteady effect outside the wake yields a negative contribution to the pressure integral. The lifting wing requires a more general consideration.

In the case of the drag of a lifting wing of span b one cannot identify a definite wake region where the effects are concentrated. Nevertheless, the Trefftz-plane (y, z plane) can be divided in a region where the integrated unsteady contribution is positive, i.e., the vertical strip $-\pi b/8 > y > \pi b/8$, and a region where the contribution is negative, i.e., $-\pi b/8 > y$ and $\pi b/8 < y$.

The essential difference between the treatments of the actuator force and the viscous force and the treatment of the drag of a lifting wing is found in the blockage effect. The blockage effect of the lifting wing is only formally present because the wake is not concentrated within boundaries. As a confusing aspect the low pressure in the Trefftz-plane due to the blockage is masked by the net high pressure due to the lift.

In the models considered it is shown, except the lifting wing model where it is postulated, that an external force is balanced by unsteady pressure forces on the end faces of an overall control volume. These unsteady pressures are either supported by boundaries or they accelerate the overall mass in the control volume. This conclusion is valid in a steady as well as in an unsteady formulation.

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